Perception, 1982, volume 11, pages 325-335

Visual perception of relative mass in dynamic events

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Received 25 March 1981, in revised form 20 November 1981

Abstract. Two experiments are reported which examine how the *dynamic* property of relative mass in collision events is specified by *kinematic* properties of a visual motion display. In most cases observers are accurate at detecting the 'heavier' of two objects, although they do not take advantage of the completely general optic information that is available. Instead, they rely on limited information that breaks down at extreme values of elasticity and relative initial velocity. In addition, observers appear to utilize different information for relative mass with different types of collisions. It is suggested that reliance on such limited information may be appropriate for perceivers operating in the restricted context of a terrestrial environment.

1 Introduction

As any viewer of animated cartoons will attest, visual displays that produce changes in optic structure can yield a variety of distinct classes of perceived events. Whether the display elements are points of light or contoured forms, their optic motions seem to result in a limited number of perceptual effects. As Johansson and his coworkers (Johansson 1950, 1974; Johansson and Jansson 1968) have shown, elements with common vector components are observed to be physically linked, and such motion subsystems can be hierarchically nested (see also Restle 1979). Other displays contain elements that appear to interact causally or dynamically with one another, as in Michotte's (1946/1963) collision experiments. Effects of animate or biological motion can also be obtained, as with point-light walkers (Johansson 1973; Cutting et al 1978), Michotte's (1946/1963) example of caterpillar locomotion, and the simulation of human growth (Pittenger and Shaw 1975; Todd et al 1980). Such moving elements may even appear to perform intentional, goal-directed acts, as shown by Heider and Simmel's (1944) famous film of geometric shapes fighting, fleeing, and courting. All of these motion effects are perceptually compelling and highly stable over observers and viewing occasions, despite the viewer's awareness that the elements are merely lights on a screen (for a review see Johansson et al 1980).

Perceptual theory is thus confronted with the problem of accounting for these nonarbitrary regularities in the perception of motion displays. Unlike material objects and organisms, images on a screen have no substance, animacy, or intentions of their own, and do not causally interact with one another. Yet observers of such displays consistently perceive these classes of real-world events. The distinction made in classical physics between kinematics and dynamics is useful in this regard. *Kinematics* is the branch of mechanics that deals with aspects of pure motion (position, velocity, acceleration, jerk, etc) without reference to the masses or forces involved therein. *Dynamics*, in contrast, is the branch of mechanics that deals with the motions and equilibria of systems by taking into account the various forces to which they are subjected (and the related properties of mass, friction, elasticity, viscosity, etc). Whereas the motion of a physical object can be described in either kinematic or dynamic terms, the motion of a visual image can only be described kinematically, since it has no material substance and cannot be affected by physical force. The theoretical problem, therefore, is to account for the perception of dynamic (and biological and intentional) properties in a visual display that, by definition, can only exhibit kinematic properties.

The traditional approach to this problem is to assume that the observer attributes dynamic properties to a display with the aid of cognitive processes. An important function of cognition, according to this view, is to elaborate impoverished visual information by forming hypotheses or inferences about real-world events that would be most likely to produce an observed pattern of stimulation. The central problem with this approach is that it does not satisfactorily explain how such hypotheses are generated (see Shaw et al 1981)—why should the perceptual distinctions between animate and inanimate, or between causally-related and independent, fall where they so regularly do?

An alternative approach to the problem of event perception suggested by Runeson (1977a; see also Johansson 1978) is to assume that the dynamic properties of real objects are specified by the kinematic properties of their images on an optic projection surface. The primary evidence in support of this view is simply that perception is seldom capricious. The research cited above has demonstrated that the perception of dynamic, biological, and intentional properties of events is predictably determined by the nature of the display itself (see also Johansson et al 1980); specifically, by constraints on the patterns of optic motion that are in keeping with what is known about the regularities of natural events in the world. Natsoulas (1960, 1961) and Runeson (1977a), for example, have attempted to account for the perception of Michotte's collision events by examining how real-object motions are governed by the laws of physics.

The research reported in the present paper is an extension of Runeson's (1977a) analysis of collision events. Its goals are twofold: first, to demonstrate mathematically that the dynamic property of relative mass is indeed specified by the kinematic properties of a visual display; and second, to demonstrate empirically that human observers are able to make use of the visual information that is available.

2 Experiment 1: perception of relative mass

In order to demonstrate how a visual display could provide information about dynamic properties of events it is useful to consider a specific example borrowed from Runeson (1977a). Suppose that there are two objects, **a** and **b**, that undergo a head-on collision. Let u_a and u_b be their respective velocities before the collision, v_a and v_b be their respective velocities after the collision, and m_a and m_b be their respective masses. We know from the law of conservation of momentum that

$$\frac{m_a}{m_b} = \frac{u_b - v_b}{v_a - u_a} .$$

In other words, the relative masses of the two objects can be uniquely determined from their velocities before and after collision (see the appendix). This relationship is completely general and valid in any frame of reference, including a projection onto a two-dimensional display screen. Here the dynamic property of relative mass is optically specified by an expression made up of four kinematic terms.

Another dynamic property of relevance to collision events is elasticity. The amount of elasticity, e, sometimes called the coefficient of restitution, is a constant for any given pair of objects, and is determined empirically by the following equation:

$$e = \frac{v_a - v_b}{u_b - u_a} .$$

The value of e may vary between 0 and 1 for different pairs of objects, 1 corresponding to a perfectly elastic collision (ie no kinetic energy is lost), and 0 corresponding to a perfectly inelastic or fully damped collision. For example, tests have shown that for a superball bouncing on a hardwood floor $e = 0{\cdot}89$; for a bouncing basketball $e = 0{\cdot}76$; and for a bouncing softball $e = 0{\cdot}32$ (Hay 1973).

An important implication of the preceding analysis is that, if one were allowed to observe a collision between two objects, one could in principle determine accurately their relative mass and elasticity. In addition, the judgments of relative mass should be independent of elasticity, and vice versa. Experiment 1 is an attempt to test this *momentum hypothesis* for the perception of relative mass, to determine whether human observers, without formal training in physics, are able to take advantage of this available information.

2.1 *Method*

Head-on collisions between two horizontally moving objects were generated with the aid of a Nova II computer with a Tektronix 611 fast phosphor display unit. The collision events and their parameters are illustrated in figure 1. Two squares appeared from off screen, collided without deformation and with an instantaneous change in velocity, and rebounded until one object went back off screen or the subject responded. The displays were updated at a rate of approximately 21 frames s^{-1} and the events lasted from 1·0 to 1·5 s. The screen was approximately 2 ft from the observer, subtending 20 deg of visual angle, and each square was approximately 1 deg in diameter.

The value of relative mass, m_a/m_b, was selected randomly on each trial from possible values of 1·25, 1·50, 2·0, and 3·0; and the value of elasticity was selected at random from possible values of 0·9, 0·5, and 0·1. These randomizations were constrained such that there were twenty trials for each possible combination of relative mass and elasticity. The right/left position of the computationally heavier object was selected randomly.

The initial screen velocity of the left-hand object was selected randomly from possible values of 0·75, 0·80, 0·85, ..., 1·75 in s^{-1} (1·8 to 4·2 deg s^{-1}). The initial velocity of the right-hand object was determined by subtracting 2·5 from the velocity of the left-hand object, producing a range of values between $-0{\cdot}75$ and $-1{\cdot}75$ in s^{-1}. The velocities of the objects after collision were computed from equations (3) and (4) in the appendix (see Runeson 1977a). No sliding friction was calculated into the motion.

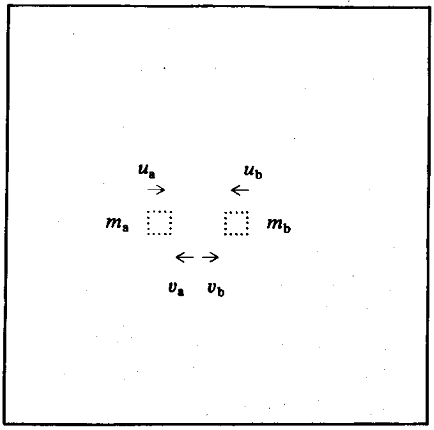

Figure 1. Display parameters for collision events: m = mass, u = velocity before collision, v = velocity after collision.

Sixteen naive observers participated in the experiment to fulfill the requirements of an introductory psychology course. None of the observers reported that he/she had more than a casual knowledge of physics. Subjects were run individually, seated in front of the display screen. They were instructed to press one of two response buttons on each trial to indicate which of the two objects looked heavier. None of the observers had difficulty in understanding these instructions, and all of them reported that the task seemed quite natural. An experimental session lasted about 20 min, with twelve practice trials followed by 240 test trials. No feedback was provided on any trial. Data on elasticity, relative mass, and the subject's response were recorded by the computer.

2.2 *Results and discussion*

Figure 2 shows the percentage of correct responses for each combination of elasticity and relative mass. The level of performance is above chance in all conditions, indicating that human observers are able to make use of information about the relative mass of two colliding objects.

There is, however, a noticeable effect of elasticity, with a decline in performance particularly at the lowest value of $e = 0{\cdot}1$. This suggests that observers are insensitive to the general momentum information about relative mass, since that information is independent of elasticity. Several of the observers reported in a debriefing session that their responses were based on the relative speeds of the two objects after collision—the slower object appearing heavier. When the observers were pressed to justify this approach, none was able to provide a coherent explanation beyond saying that the slower object looked heavier. A subsequent analysis of the data revealed that the object moving more slowly after collision was judged to be heavier on 92·5% of the trials. As figure 2 shows, reliance on this information yields reliable judgments except at small values of elasticity and relative mass.

A mathematical analysis of this *final-speed hypothesis* is given in the appendix. The analysis shows that when two objects approach each other at the same speed their relative speeds after collision correspond perfectly with their relative masses. This information is limited in its generality, however, since the correspondence is diminished as the initial approach speeds become unequal and as e becomes less than

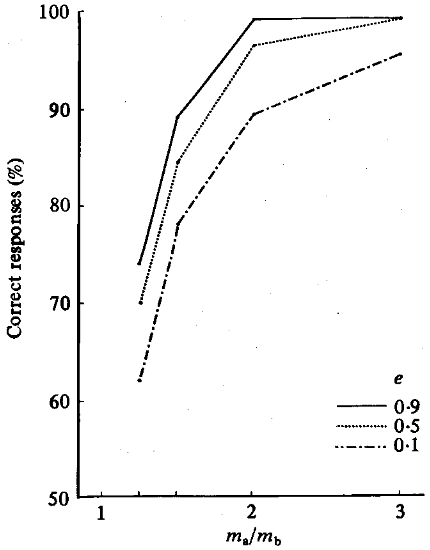

Figure 2. Percentage of correct responses for each combination of relative mass, m_a/m_b, and elasticity e in experiment 1. Each point represents the mean of 320 judgments, $N = 16$.

1 (except when $u_a = u_b$). It is reduced to zero for perfectly inelastic collisions when one of the objects is initially stationary.

Runeson (personal communication) has pointed out that there is other available information for relative mass that is more appropriate when a moving object collides with a stationary object. For example, consider a perfectly elastic collision between moving object **a** and stationary object **b**, assuming that the initial velocity of a, u_a, is positive (say, in a rightward direction). If the two objects have the same mass ($m_a = m_b$), then the final velocity of **a** will be zero and the final velocity of **b** will be equal to the initial velocity of **a** ($v_a = 0$ and $v_b = u_a$)—a complete transfer of momentum. On the other hand, if m_a is greater than m_b, then v_a will be positive (ie **a** will continue moving rightward after collision) and v_b will be greater than u_a. If m_a is less than m_b, then v_a will be negative (ie **a** will rebound leftward) and v_b will be less than u_a.

These relationships provide two more hypotheses of useful information for relative mass, analyzed in the appendix: the *direction hypothesis* involving the direction of motion of object **a** after collision, and the *initial/final speed hypothesis* involving the speed of object **a** before collision relative to the speed of object **b** after collision. Use of these sources of information when **b** is not initially stationary or when $e < 1$, however, would produce characteristic patterns of errors (see the appendix and tables 1 and 2).

Since the general momentum hypothesis did not account for the result of experiment 1, experiment 2 was designed to compare the latter three hypotheses of limited information in several specific contexts.

3 Experiment 2: information for relative mass

The three hypotheses about information for relative mass compared in experiment 2 can be summarized in the following way. The final-speed hypothesis can be stated in the form of a biconditional about the judgment of relative mass:

$$m_a > m_b \longleftrightarrow |v_a| < |v_b| ;$$

the direction hypothesis can be similarly stated as

$$m_a > m_b \longleftrightarrow v_a > 0 ;$$

and the initial/final speed hypothesis can be stated as

$$m_a > m_b \longleftrightarrow v_b > u_a .$$

3.1 *Methods*

The apparatus and general procedure were similar to those used in experiment 1, but two conditions were run in separate blocks of one hundred and twenty trials each. In the *moving condition* two square objects appeared on opposite sides of the display screen, approached each other with equal speeds of 1·25 in s^{-1}, collided without deformation, and rebounded until one object went back off screen or the subject responded. In the *stationary condition* moving object **a** appeared on the left and collided with stationary object **b** in the center of the display screen. In both conditions the possible values of elasticity were 0·1, 0·5, and 0·9; and the possible values of relative mass, m_a/m_b, were 0·33, 0·50, 0·67, 0·80, 1·25, 1·50, 2·0, and 3·0. Each combination of elasticity and relative mass occurred five times in each of the two conditions, and the trials were arranged in random order.

Sixteen naive observers participated in the experiment to fulfill the requirements of an introductory psychology course. As in the previous experiment, the observers had no more than a casual knowledge of physics. Half of the observers were presented with the stationary condition followed by the moving condition, while the other half received the reverse order.

3.2 *Results and discussion*

The results predicted by the three hypotheses are compared with the actual data in tables 1 and 2. An analysis of variance revealed that the effect of order and the order-by-treatment interaction were not significant. (The sums of squares for the order-by-treatment interaction were pooled into the error term for testing the treatment effects.)

In the *moving condition* (table 1) the observers responded correctly on over 95% of the trials. The effect of elasticity, the effect of relative mass, and the interaction between the two were all negligible. These results suggest that the observers utilized

Table 1. Percentage of correct responses predicted by mass hypotheses and actual data from experiment 2: moving condition.

e	Hypothesis[a]	Relative mass, m_a/m_b								
		0·33	0·50	0·67	0·80	1·25	1·50	2·0	3·0	$\bar{X}$
0·9	FS	100	100	100	100	100	100	100	100	100
	D	100	100	100	100	0	0	0	50	56
	I/FS	100	100	100	100	100	100	100	100	100
	Data	**99**	**95**	**95**	**89**	**90**	**98**	**96**	**98**	**95**
0·5	FS	100	100	100	100	100	100	100	100	100
	D	100	100	100	100	0	0	50	100	69
	I/FS	100	100	100	100	0	0	50	100	69
	Data	**99**	**95**	**95**	**91**	**85**	**94**	**95**	**98**	**94**
0·1	FS	100	100	100	100	100	100	100	100	100
	D	100	100	100	100	100	100	100	100	100
	I/FS	100	100	100	100	0	0	0	0	50
	Data	**94**	**99**	**98**	**95**	**91**	**96**	**100**	**99**	**96**

[a] FS, final speed; D, direction; I/FS, initial/final speed.

Table 2. Percentage of correct responses predicted by mass hypotheses and actual data from experiment 2: stationary condition.

e	Hypothesis[a]	Relative mass, m_a/m_b								
		0·33	0·50	0·67	0·80	1·25	1·50	2·0	3·0	$\bar{X}$
0·9	FS	0	0	0	0	100	100	100	100	50
	D	100	100	100	100	100	100	100	100	100
	I/FS	100	100	100	100	100	100	100	100	100
	Data	**90**	**75**	**48**	**38**	**93**	**96**	**95**	**96**	**79**
0·5	FS	0	0	0	0	100	100	100	100	50
	D	100	50	0	0	100	100	100	100	69
	I/FS	100	100	100	100	0	0	50	100	69
	Data	**74**	**43**	**23**	**24**	**95**	**94**	**96**	**96**	**68**
0·1	FS	0	0	0	0	100	100	100	100	50
	D	0	0	0	0	100	100	100	100	50
	I/FS	100	100	100	100	0	0	0	0	50
	Data	**56**	**45**	**36**	**39**	**65**	**76**	**68**	**78**	**58**

[a] FS, final speed; D, direction; I/FS, initial/final speed.

the property of relative final speed, as in experiment 1. Had they used direction or initial/final speed, the number of errors would have been considerably higher, and the statistical effects of elasticity, relative mass, and their interaction would have been significant.

In the *stationary condition* (table 2) the overall level of performance was 68·2% correct, considerably lower than in the *moving condition* ($F_{1,705} = 307{\cdot}6, p < 0{\cdot}001$), with the majority of errors occurring when the stationary object was heavier ($F_{1,705} = 314{\cdot}9, p < 0{\cdot}001$). This latter finding is the opposite of what would be expected if observers had relied on initial/final speed. When the moving object was heavier ($m_a/m_b > 1$), observers responded correctly on 87·3% of the trials. The effect of elasticity was significant ($F_{2,705} = 26{\cdot}65, p < 0{\cdot}001$), but the effect of relative mass and the interaction of the two were negligible. With the exception of the elasticity effect, these results are consistent with both the final-speed and direction hypotheses.

On the other hand, when the stationary object was heavier ($m_a/m_b < 1$), observers responded correctly on only 49·1% of the trials. There was a significant effect of elasticity ($F_{2,705} = 19{\cdot}87, p < 0{\cdot}001$), a significant effect of relative mass ($F_{3,705} = 37{\cdot}7$, $p < 0{\cdot}001$, and a significant interaction ($F_{6,705} = 3{\cdot}79, p < 0{\cdot}005$). These results are completely incompatible with both the final-speed and the initial/final speed hypotheses. If observers had relied on the latter, they would have responded correctly on every trial when the stationary object was heavier, and if they had relied on the former, they would have responded incorrectly on every trial (see the appendix). Although all three significant effects are consistent with the direction hypothesis, only 62% of the responses were predicted by the use of this property.

4 General discussion

There are three major conclusions that can be drawn from these results. First, observers are able to judge the relative mass of two colliding objects without special training or feedback. Second, the information on which these judgments are based is limited, being valid only in certain contexts and breaking down at low values of elasticity, despite the fact that there is other more complex information available that is perfectly general. Third, the evidence suggests that observers use different information in different contexts. When two moving objects collide, observers base their responses on the relative speeds of the objects after the collision. When one of the objects is initially stationary, observers rely on some other as yet undetermined property that leads to errors only when the stationary object is heavier, and that is significantly affected by elasticity and the difference in mass.

It is interesting to note that other researchers have also found tasks in which observers rely on sources of information that are valid only in specific contexts. Braunstein (1976) has used the term 'heuristic' to describe an observer's performance in these tasks. The concept is borrowed from computer science and refers to a computational procedure that usually produces desirable results, but may produce undesirable results in unusual circumstances. The concept of an algorithm, in contrast, refers to a procedure that guarantees desirable results in all possible circumstances(1). The advantage of heuristics is that they generally have a lower computational cost than algorithms, being performed in less time or with a smaller number of processing elements. Thus, a heuristic is often superior to an algorithm

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(1) The concepts of heuristic and algorithm can be used in two distinct ways: (i) as formal descriptions of the behavior of a system (eg Shaw and Todd 1980), or (ii) as entities internal to a system that control the behavior of the system (eg Ullman 1980). The former interpretation is the one intended in the present discussion. We do not wish to suggest that observers refer to an internally represented set of rules in order to judge the relative mass of two colliding objects.

when the boundary conditions of a problem are subject to certain restrictions or constraints. In many instances the possibility of an occasional error is a small price to pay for an increase in speed or efficiency. The results of the present experiments are supportive of Braunstein's description of perceptual performance. Although a reasonably simple general algorithm exists for determining the relative mass of any two colliding objects [see appendix, equation (3)], an even simpler heuristic apparently provides a better description of human observers' behavior, as when judgments of relative mass are based on relative final-speed information.

What is the advantage of limited information in this particular task? For a computer-science application this type of question would be answered by a cost-benefit analysis. Within a given context of constraint the expected costs of inappropriate responses would be compared with the expected savings in processing efficiency. However, this type of analysis is difficult to perform when dealing with biological systems. To specify adequately the constraint for perceiving the relative mass of two colliding objects it would be necessary to determine the possible values of velocity, mass, and elasticity for all of the different objects in an individual's natural environment. To specify the cost of inappropriate responses it would be necessary to determine how relative mass judgments could significantly affect an individual's well-being. And to specify the potential savings in efficiency it would be necessary to determine the evolutionary or metabolic costs of the ability to detect different kinematic relationships in a visual display. In this case, 'heuristic' behavior would be a consequence of the constraints prevailing within the natural context, which make a more costly general 'algorithmic' solution unnecessary and undesirable. When taken outside of this context, however, the system may function poorly, since it is asked to perform a task it was not designed to perform (see Gibson 1966, 1979; Runeson 1977b). This approach cautions against conclusions about perceptual error and illusion without consideration of the natural constraints upon perceptual devices (see Shaw and Cutting 1980; Shaw et al 1981).

In the absence of such an analysis, however, there is no good reason to expect that human observers would exhibit the seemingly arcane ability to perceive relative mass in collision events. Although collisions are reasonably common in a terrestrial environment, it is difficult to imagine naturally occurring situations in which it is important for an individual to make relative-mass judgments about colliding objects. In contrast, Runeson and Frykholm (1981) have recently obtained accurate judgments of object mass when subjects observed videotapes of a person lifting the object, both under full illumination and with point-lights on the actor's joints. Mass judgments might be more frequently required in natural situations similar to this, in which an object is about to be handed or thrown to the observer, than they are in the case of collisions. Nevertheless, none of the observers in the present experiments experienced any difficulty in performing the task without the benefit of training or feedback. All reported that the relative 'heaviness' of the objects was clearly evident in the displays, especially in the moving condition. This high degree of perceptual salience is also revealed by the absence of individual differences. Indeed, for most of the cells in tables 1 and 2 the percentage of correct responses was approximately the same for every observer.

The present results and the related findings of Runeson and Frykholm provide convincing evidence that human observers are indeed sensitive to kinematic visual information about the mass of an object. How this information is actually used in more natural contexts is a problem that calls for further research.

Acknowledgments. This research was supported by a grant from the University of Connecticut Research Foundation (0215-35-510) and by a National Science Foundation graduate fellowship to the second author. The authors wish to thank Sverker Runeson, Robert Shaw, and Michael Turvey for their comments on an earlier draft.

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APPENDIX

Assume that there are two colliding objects, **a** and **b**, with velocities before collision of u_a and u_b, respectively, such that

$$u_b = ku_a. \tag{1}$$

Their respective velocities after collision are v_a and v_b; and their respective masses are $m_a = pm$ and $m_b = qm$, where $m = m_a + m_b$ is the total mass of the two objects in combination, so that

$$p + q = 1. \tag{2}$$

In other words, k represents relative initial velocity, and p and q represent the masses of objects **a**, and **b**, respectively expressed as proportions of the total mass.

In regard to relative mass, we know from the law of conservation of momentum that

$$\frac{p}{q} = \frac{m_a}{m_b} = \frac{u_b - v_b}{v_a - u_a}. \tag{3}$$

We also know that the coefficient of restitution, e, for two particular objects, defined as

$$e = \frac{v_a - v_b}{u_b - u_a}, \tag{4}$$

is a constant. To express the final velocities as a function of e, we rearrange equation (4) and substitute it into equation (3):

$$v_a = qu_b(1+e) + u_a(p - eq), \tag{5}$$

$$v_b = pu_a(1+e) + u_b(q - ep). \tag{6}$$

To eliminate u_b and q, we substitute equations (1) and (2) into equations (5) and (6), and rearrange the terms:

$$v_a = u_a[p(1+e)(1-k) + k(1+e) - e], \tag{7}$$

$$v_b = u_a[p(1+e)(1-k) + k]. \tag{8}$$

These equations can be used to derive three different optic properties that correspond to relative mass with varying degrees of generality.

(a) *Final-speed hypothesis*: Observers will respond that m_a is greater than m_b if and only if $|v_a|$ is less than $|v_b|$.

It is assumed that object **a** is initially moving in a rightward direction ($u_a > 0$). First, if $v_a > 0$ (rightward after collision), then from equation (7) by this hypothesis the observer will respond that $m_a > m_b$ whenever

$$u_a[p(1+e)(1-k) + k(1+e) - e] > 0. \tag{9}$$

Since we have assumed that $u_a > 0$, this reduces to

$$p > \frac{[e/(1+e)] - k}{1-k}. \tag{10}$$

This response will be incorrect when p is between 0·5 and the value given by equation (10) for a certain combination of e and k. Since $|v_a|$ cannot be greater than $|v_b|$ (without **a** 'passing through' **b**), by this hypothesis judgments can only err in the direction of object a (ie **a** judged heavier than **b**).

If, on the other hand, $v_a < 0$ (leftward after collision), then $|v_a| = -v_a$, and from equations (7) and (8) by this hypothesis the observer will respond that $m_a > m_b$ whenever

$$u_a[p(1+e)(1-k) + k] > -u_a[p(1+e)(1-k) + k(1+e) - e], \tag{11}$$

which reduces to

$$p > r, \qquad r = \frac{e - [2k/(1-k)]}{2(1+e)} \tag{12}$$

This response will be incorrect when p is between 0·5 and r. If $0 \cdot 5 < p < r$ the observer will incorrectly respond that $m_a < m_b$, but since we assume that $-1 \leqslant k \leqslant 0$ in experiment 2, such cases cannot occur. Alternately, if $r < p < 0 \cdot 5$, the observer will incorrectly respond that $m_a > m_b$.

Hence, under the conditions of experiment 2 observers can only err in the direction of object a, according to the final-speed hypothesis. Relative final speed corresponds completely to relative mass when two objects approach each other at the same speed ($k = -1$), regardless of e.

(b) *Direction hypothesis*: Observers will respond that m_a is greater than m_b if and only if v_a is positive.

Assuming that object b is initially stationary ($k = 0$), then equation (10) for $v_a > 0$ reduces to

$$p > \frac{e}{1+e}, \tag{13}$$

and equation (12) for $v_a < 0$ reduces to

$$p > \frac{e}{2(1+e)}. \tag{14}$$

Observers will respond that $m_a > m_b$ when these conditions hold. This response will be incorrect when p is between 0·5 and the value given by equation (13) or equation (14). If, for example, $e/(1+e) < p < 0 \cdot 5$, the observer will incorrectly respond that $m_a > m_b$. Alternately, if $0 \cdot 5 < p < e/(1+e)$, the observer will incorrectly respond that $m_a < m_b$, but, since in practice e must be between 0 and 1, $e/(1+e)$ can never be greater than 0·5 and hence such cases cannot occur. Since the same obviously holds for equation (14), by this hypothesis judgments can only err in the direction of object a. Final direction corresponds completely when $k = 0$ and $e = 1$, or when $k = -1$ and $e = 0$.

(c) *Initial/final speed hypothesis*: Observers will respond that m_a is greater than m_b if and only if v_b is greater than u_a.

If $v_b > u_a$, then from equation (8) by this hypothesis observers will respond that $m_a > m_b$ whenever

$$u_a[p(1+e)(1-k)+k] > u_a, \tag{15}$$

which reduces to

$$p > \frac{1}{1+e}. \tag{16}$$

Thus, the correspondence of this property is unaffected by the velocities of the two objects before collision, but it is reduced if $e < 1$. The observer will respond incorrectly that $m_a < m_b$ when $0 \cdot 5 < p < 1/(e+1)$. Conversely, the observer will incorrectly respond that $m_a > m_b$ when $1/(1+e) < p < 0 \cdot 5$, but such cases cannot occur in practice since $1/(1+e)$ can never be greater than 0·5. Hence, judgments based on this property can only err in the direction of object **b**.

Predictions of subject performance based on these equations are given in tables 1 and 2 for each hypothesis.