Research Article

ON THE AFFINE STRUCTURE OF PERCEPTUAL SPACE

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Abstract—*Affine geometry is a generalization of Euclidean geometry in which distance can be scaled along parallel directions, though relative distances in different directions may be incommensurable. This article presents a new procedure for testing the intrinsic affine structure of a psychological space by having subjects perform bisection judgments over multiple directions. If those judgments are internally consistent with one another, they must satisfy a theorem first proved by Pierre Varignon around 300 years ago. In the experiment reported here, this procedure was employed to measure the perceived structure of a visual ground surface. The results revealed that observers' judgments were systematically distorted relative to the physical environment, but that the judged bisections in different directions had an internally consistent affine structure. Implications of these findings for other possible response tasks are considered.*

One of the most popular ways of modeling perceptual (or cognitive) phenomena is to represent percepts (or concepts) as points within some psychological space. Often it is assumed, moreover, that a psychological space has an underlying geometric structure that constrains how different judgments are related to one another. There are many different geometries that could potentially be used for modeling psychological phenomena (e.g., see Suppes, 1977; Suppes, Krantz, Luce, & Tversky, 1989). In the present article, we consider one called affine geometry, which may be unfamiliar to many researchers in psychology. We first discuss why this particular geometry might be useful, and we then introduce a new technique for testing whether the affine properties of a psychological space are internally consistent.

Of all the possible aspects of geometric structure, the one that has attracted the most attention in the psychological literature is called a distance metric. The metric of a space is a mathematical function that relates distances in different directions. For example, the metric of Euclidean space is defined by the Pythagorean theorem. Euclidean geometry was developed more than 2,000 years ago as an abstract model of the physical environment, but it does not always provide an adequate description of the distance relations within a psychological space. Other possible metrics that have been proposed in this context include Minkowski metrics (e.g., see Attneave, 1950) and Riemannian metrics (e.g., see Luneberg, 1947), both of which are based on more general conceptions of space that need not be Euclidean.

For any space to be considered as metric, there must be a distance function δ by which every pair of points can be assigned a nonnegative distance value that conforms with three basic axioms:

Minimality:	$\delta(a,b) \geq \delta(a,a) = 0$
Symmetry:	$\delta(a,b) = \delta(b,a)$
Triangle inequality:	$\delta(a,b) + \delta(b,c) \geq \delta(a,c)$

It is interesting to consider, therefore, the extent to which these axioms are satisfied for various types of psychological judgments. Shepard (1964) made an early attempt to address this issue. He asked subjects to make similarity judgments for line drawings of a circle with a single radial line; the size of the circle and the orientation of the line could be varied systematically. His results revealed that the observers' responses could not be adequately characterized by any static metric, because the relative weighting of the different dimensions in the overall similarity judgment could change with a subject's state of attention. When responses obtained with different states of attention were combined, the resulting data contained clear violations of the triangle inequality.

To provide a more intuitive example of how this could occur, Shepard offered the following *Gedanken* experiment: Suppose that subjects were asked to rate the dissimilarity (i.e., distance) between words on a scale from 1 to 100. If given the pair *table* and *fable*, they would likely notice that the words sound alike, and assign a low dissimilarity rating, such as 15. If given the words *table* and *chair*, the subjects would probably switch their attention to the frequent co-occurrence of these objects in the natural environment, and would again provide a low dissimilarity rating, such as 10. However, *chair* and *fable* have no apparent resemblance at all, and would likely be given a high dissimilarity rating, such as 80. This pattern of results would be a violation of the triangle inequality, and would therefore indicate that the psychological representation of these words is nonmetric.

Let us now consider how affine geometry might provide a more appropriate model of these phenomena. Affine geometry is a generalization of Euclidean geometry, with a more limited set of assumptions. Both geometries share the axioms of incidence and the parallel postulate, but affine geometry does not require the axioms of congruence. (See Blumenthal, 1961, for a more detailed discussion of the axiomatic bases of alternative geometries.) Within an affine space, it is possible to compare the relative lengths of all parallel line segments,[1] though the relative lengths of nonparallel line segments may be incommensurable.

The psychological distinction between parallel and nonparallel distance intervals was nicely demonstrated in another experiment reported by Shepard (1964). He asked observers to perform two tasks, one that compared intervals along a single dimension, and another that compared intervals across different dimensions. On the intradimen-

1. It is possible to define an affine geometry that is based only on the axioms of incidence and Euclid's parallel postulate and that does not provide sufficient structure to establish an equivalence relation between parallel line segments. In order to ensure this property, it is necessary to include an additional axiom, which can take the form of either Desargues's theorem or Pappus's theorem (see Blumenthal, 1961).

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sional task, observers judged the difference in orientation (i.e., angle) between pairs of line segments that were connected at one end. One pair of lines was presented as a standard, and observers could adjust the relative orientation of a second test pair so that the angle between them appeared equal to the standard angle. On the cross-dimensional task, observers were required to adjust an angle to match the size difference between two circles. All subjects agreed that the intradimensional task was quite trivial, but most of them insisted that they could find no intuitive basis for performing the cross-dimensional judgments. Consistent with these subjective reports, the variances obtained for the cross-dimensional task were 19 times larger than the variances for the intradimensional task, in which the matching stimuli differed along a single dimension.

Shepard (1964) argued that cross-dimensional confusions are likely to be limited to artificially defined spaces, whose component dimensions have no natural relationship with one another. However, more recent evidence has revealed that similar distinctions between judgments of parallel and nonparallel distance intervals can also occur for other psychological spaces that are generally believed to be homogeneous. For example, when observers are asked to compare distance intervals in the physical environment, their judgments are more accurate and reliable when the comparison stimuli are parallel to one another than when they are not (Norman, Todd, Perotti, & Tittle, 1996; Todd & Bressan, 1990). These findings provide strong evidence that affine properties of the physical environment may be more perceptually salient than its metrical properties.

It is important to keep in mind when considering these issues that there are many possible geometries involving different sets of underlying assumptions that could potentially be useful within mathematical psychology. Faced with this plethora of possibilities, how is one to decide which geometry is most appropriate in any given context? One could, of course, just assume a particular geometry and hope that its underlying assumptions are valid, but we believe that a better approach is to perform an independent check of those assumptions, such as Shepard's (1964) test of the triangle inequality. In order to adopt this approach, however, it would be necessary to devise a set of formal procedures for assessing the internal consistency of observers' judgments about the relevant properties of each geometry to be considered.

One way of testing the internal consistency of affine judgments is to exploit a theorem that was first proven by Pierre Varignon around 1700. Let P_1, P_2, P_3, and P_4 be arbitrarily selected points that define the vertices of a quadrilateral. Let Q_1, Q_2, Q_3, and Q_4 be the bisection points along the four quadrilateral edges, respectively, and let T_1 and T_2 bisect the intervals between the bisections of opposing edges (see Fig. 1). In an affine space, the points Q_1 through Q_4 will form a parallelogram, and the points T_1 and T_2 will be coincident with one another. It is important to recognize that intersecting line segments do not generically bisect one another. Thus, the coincidence of T_1 and T_2 in Varignon's theorem imposes on the structure of an affine space a severe constraint by which interval bisections in different directions are formally related to one another.

Varignon's theorem also provides a straightforward procedure for measuring the internal consistency of affine structure for any psychological space in which it is possible to make bisection judgments. The procedure involves two separate phases. Subjects would first make bisection judgments for all edges of an arbitrary quadrilateral to obtain Q_1 through Q_4. Next they would bisect the intervals between the judged bisections of the opposing edges to obtain T_1 and T_2. If the space is affine, then the final two judgments must be statistically equivalent.

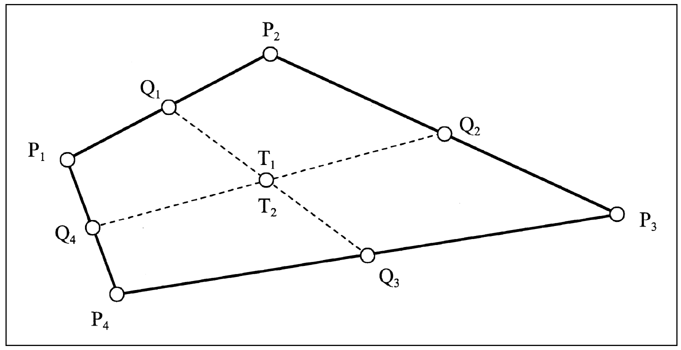

Fig. 1. A Varignon configuration similar to the configurations used in the present experiment to investigate the affine structure of perceptual space. Points P_1 through P_4 mark the vertices of a quadrilateral. Points Q_1 through Q_4 bisect the edges of the quadrilateral, and points T_1 and T_2 bisect the intervals between the opposing edge bisections, Q_1Q_3 and Q_2Q_4.

Recently, we have employed this procedure to measure the intrinsic affine structure of a perceived ground surface. This particular type of psychological space is of special interest because prior research has shown that it is systematically distorted relative to the actual physical environment (e.g., see Battro, Netto, & Rozestraten, 1976; Koenderink, van Doorn, & Lappin, 2000; Norman et al., 1996; Wagemans & Tibau, 1999). That is to say, physically straight lines can appear perceptually to be curved, and intervals of equal length can appear perceptually to be unequal. These findings demonstrate that observers' judgments of affine properties in the environment can be physically inaccurate, but are they internally consistent? Our research was designed to address this question.

METHOD

Apparatus

The stimuli were created and displayed on a Macintosh G3 computer with a 21-in. monitor. The displays were viewed through LCD (liquid crystal display) shuttered glasses that were synchronized with the monitor's refresh rate. The different views of a stereo pair were displayed at the same position on the monitor screen, but they were temporally offset. The left and right lenses of the LCD glasses shuttered synchronously with the display at an alternation rate of 60 Hz, so that each of the two stereo images could be seen only by the appropriate eye. When operating in stereo mode, the monitor had a spatial resolution of 1280 × 484 pixels. The displays were viewed from a distance of 57.3 cm so that each image subtended a visual angle of 38.5° × 25.8°. Head movements were restricted using a chin rest.

Stimuli

Each display depicted a blue and black textured ground surface with three vertical red posts (see Fig. 2). The simulated ground surface was constructed from a 6.0- × 6.0-m rectangle located 15.0 cm below the point of observation. The right and left edges of this rectangle were located just outside the viewing frustum of the display window so that they would not be visible. The surface texture was created from a random check pattern that was blurred and then thresholded to pro-

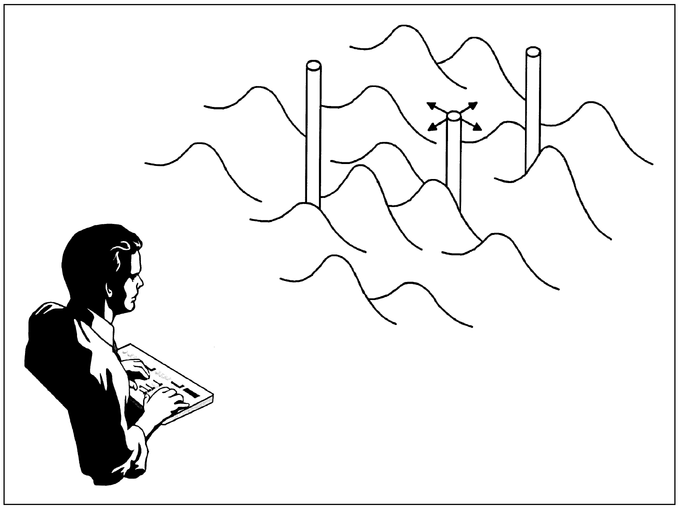

Fig. 2. A schematic view of a typical stimulus configuration used in the present experiment. Observers adjusted a vertical post on a bumpy ground surface until it appeared to bisect the interval between two other fixed posts.

duce a binary texture map. Eight different textures were created, and one was selected at random on each trial.

Two different manipulations were performed to prevent observers from judging the depth of a post from the heights of its endpoints in the visual field. First, the simulated lengths of the posts were varied over a range of 11.3 to 18.8 cm. Second, the depicted surface had a random pattern of hills and valleys, so that the height of the ground varied over a 4-cm range. A different random pattern of bumps was generated for each trial.

Two of the posts presented in each display had fixed positions on the ground that varied across trials. The position of the third post could be adjusted both horizontally and in depth by manipulating a handheld mouse. Twelve parallelogram Varignon configurations that had different sizes, shapes, and positions were created. The depths of the probe points in these configurations ranged from 131 to 397 cm, and their visual eccentricities ranged from $+16°$ to $-16°$. Each pair of probe points defined a virtual line in physical space whose length was between 26 and 134 cm; these virtual lines were slanted in depth over a range of angles between $+71°$ and $-71°$. The resulting visual angles by which the probe pairs were separated in optical space ranged from 3° to 24°.

Procedure

The task on each trial was to adjust the movable post so that it appeared to bisect an imaginary line between the two fixed posts (see Fig. 2). The bisection judgments for each configuration were obtained in two separate phases within a single experimental session. During the first phase, the positions of the fixed posts were selected from the points P_1, P_2, P_3, and P_4 of a predetermined Varignon configuration (see Fig. 1), and each pair of points was repeated on 10 separate trials. In the second phase, the fixed posts were positioned at the mean locations of the observer's judgments for Q_1 and Q_3 or for Q_2 and Q_4 obtained during the first phase. Each pair of points was again repeated on 10 separate trials to estimate the apparent locations of T_1 and T_2. Three different configurations were interleaved within each of four experimental sessions.

Observers

Six naive observers participated in the experiment and were paid $8 per hour for their services. All had normal or corrected-to-normal visual acuity. Each observer performed the sessions in a different random order.

RESULTS

Two representative patterns of responses for different observers and different configurations are shown in Figure 3. The trapezoidal boundary in each figure shows the viewing frustum of the display window. The dotted lines show the actual Varignon configuration in physical space, with depth represented in the vertical direction. The solid circular arcs show the perceived configuration as revealed by the observer's judgments. The small ellipses mark the mean positions of the observer's settings, and the axis lengths of these ellipses in different directions show the standard deviations. The two ellipses in the center of the configuration show the judged positions of T_1 and T_2, respectively.

In analyzing the errors in observers' judgments, it is useful to distinguish two separate components, which we refer to as *intrinsic* and *extrinsic*. The extrinsic errors are revealed by systematic differences between the judged test points and the positions that would be obtained if all of the bisection judgments had been performed veridically. Note in Figure 3, for example, that the judged locations of T_1 and T_2 were significantly underestimated relative to their actual locations, which provides strong evidence that the extrinsic geometry of observers' judgments was systematically distorted. As in previous studies (e.g., see Battro et al., 1976; Koenderink et al., 2000), the magnitude of these distortions and the variance of the observers' settings tended to increase with viewing distance.

Although extrinsic distortions are of considerable interest, our primary goal in the present experiment was to measure the intrinsic structure of perceptual space. Intrinsic errors are revealed in this context by violations of Varignon's theorem—that is, by the two test points of a given configuration being significantly different from one another. For both of the examples provided in Figure 3, the response distributions for the two test points were almost identical, thus indicating that the observers' judgments had an internally consistent affine structure.

Figure 4 provides a summary of the intrinsic and extrinsic errors for all 6 observers in both horizontal direction and depth. It is clear from this figure that there was a strong anisotropy in the observers' judgments. Although there were large extrinsic errors along the depth dimension, the judged horizontal positions of the test points were relatively accurate. It is also important to note that the extrinsic errors were more than four times larger than the intrinsic errors.

To analyze this pattern of results quantitatively, we performed a series of Hotelling's T^2 tests for each observer's judgments of each configuration.[2] First, we examined the extrinsic distortions of the judgments relative to the actual locations of T_1 and T_2 on the simu-

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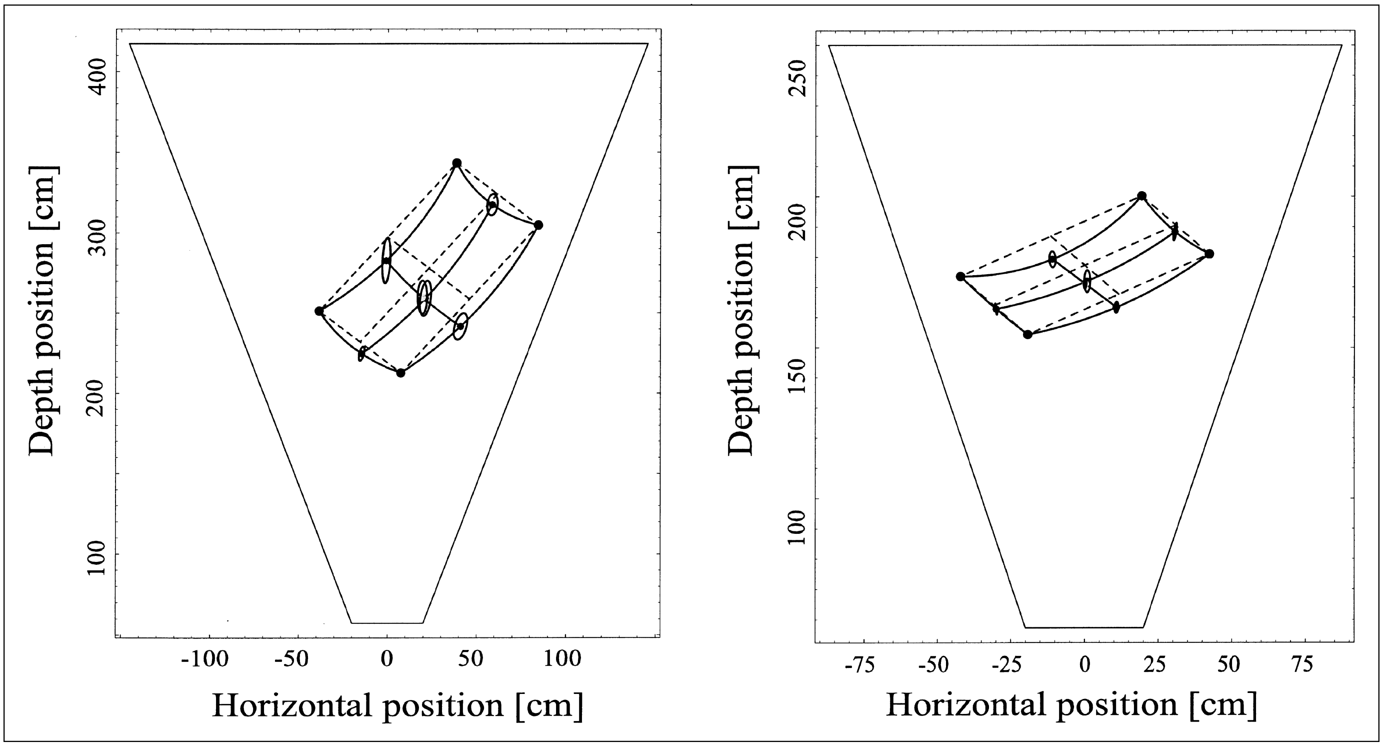

Fig. 3. Two representative patterns of responses for different observers and different configurations. The trapezoidal boundary in each panel shows the viewing frustum of the display window. The dotted lines show the actual Varignon configuration in physical space, and the solid curves show the perceived configuration as revealed by the observer's judgments. The small ellipses mark the mean positions of the observer's settings, and the axis lengths of these ellipses in different directions show the standard deviations.

lated ground surface. Out of 144 possible comparisons (6 observers × 12 configurations × 2 test points), 122 of the judged positions were significantly different from their actual locations in the depicted scene. Next, we examined the internal consistency of the observers' perceptions by comparing their settings for the two test points in each configuration. Out of 72 possible comparisons (6 observers × 12 configurations), only 6 of the judged test pairs were significantly different from one another, and they appeared to be randomly distributed among the different observers and conditions. These results suggest, therefore, that the intrinsic geometry of the observers' perceptions had an internally consistent affine structure.

DISCUSSION

It is important to recognize when evaluating the present experiment that the "geometry" of perceived space can be construed in two different ways. One possibility is to consider the extrinsic structure of observers' perceptions relative to the physical environment. From an extrinsic point of view, the structure of perceptual space (ψ) is determined by its formal relation to physical space (Φ), such that $\psi = f(\Phi)$. Within this context, the geometry of perceived space is defined by the particular set of properties that are invariant over the transformation $f(\Phi)$. Although this mapping can sometimes be affine when objects are viewed with relatively weak perspective (Koenderink & van Doorn, 1991; Koenderink, van Doorn, Kappers, & Todd, in press; Todd & Bressan, 1990; Todd & Norman, 1991), that is generally not the case

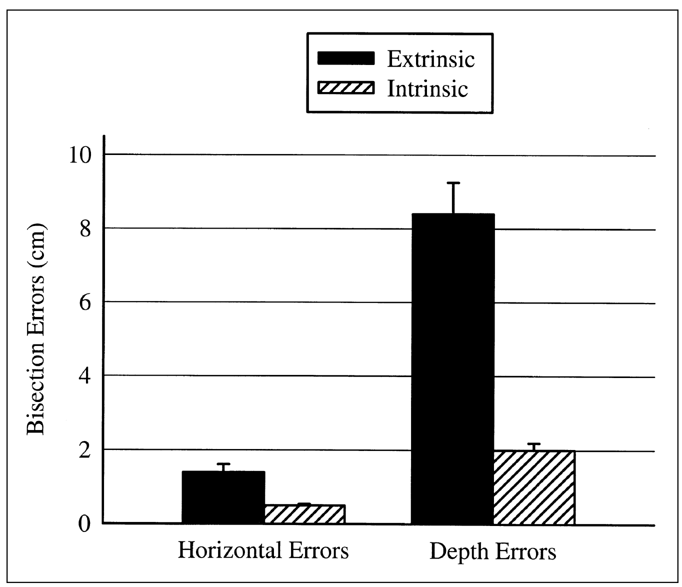

Fig. 4. Mean bisection errors in the horizontal direction and in depth averaged over all 6 observers. Intrinsic error is the average distance between the two test points (T_1 and T_2) in each configuration. Extrinsic error is the average distance between a judged test point and its actual location in physical space.

for the stereoscopic perception of extended ground surfaces. Note in Figure 3, for example, that straight lines in the physical environment can be perceived as curved, thus indicating that the transformation $f(\Phi)$ in this instance is neither affine nor projective.

In contrast to most previous studies on three-dimensional form perception, the present experiment was conducted primarily to investigate the intrinsic geometry of perceptual space. Intrinsic geometry provides a global set of constraints by which the judgments of a given observer are formally related to one another, irrespective of their relation to the external environment. Our long-term goal in this research is to develop a "toolbox" of procedures for measuring the internal consistency of observers' judgments about various aspects of geometric structure. In this case, our focus was on affine structure, and the tool for measuring its internal consistency was provided by Varignon's theorem.

Let us now consider how affine spaces are related to other types of geometric structure that have been proposed for modeling psychological phenomena. Affine geometry is based on several basic axioms, including the axioms of incidence and Euclid's parallel postulate. The axioms of incidence state that two points are connected by a single line, and that two lines are connected at a single point. The parallel postulate states that for a given line L and a given point P, there is a single line through P that is parallel to L. Euclidean geometry is a special case of affine geometry in which additional axioms are added. Thus, to demonstrate that a psychological space is affine does not preclude the possibility that it may also be Euclidean.

If a space is not affine, then it cannot be Euclidean, though it may still possess some other type of distance metric that is based on an alternative set of assumptions. For example, the distance metrics in elliptic and hyperbolic geometries are based on assumptions that specifically contradict Euclid's parallel postulate. In elliptic geometry, there are no lines through P that are parallel to L, and in hyperbolic geometry, there are an infinity of lines through P that are parallel to L. Euclidean, elliptic, and hyperbolic geometries are all special cases of a more general framework called Riemannian geometry, which can describe the structure of any smooth manifold. Riemann spaces of constant curvature—the so-called homogeneous spaces—are the only ones that allow congruence. These can be subdivided into three distinct types based on the sign of the intrinsic curvature. The geometry of spaces with no intrinsic curvature (e.g., planes or cylinders) is Euclidean; the geometry of spaces with positive curvature (e.g., spheres) is elliptic; and the geometry of spaces with negative curvature (e.g., saddles) is hyperbolic.

It is interesting to note in this regard that there have been numerous experiments that are purported to show that the intrinsic curvature of perceptual space is measurably non-Euclidean (e.g., Battro et al., 1976; Indow, 1991; Koenderink et al., 2000; Norman et al., 1996). This would seem to contradict the results of the present study, because Euclidean space is the only Riemannian geometry that is also affine. It is important to keep in mind, however, that these prior experiments have all been based on an a priori assumption that perceptual space has a stable Riemannian distance metric, but there is no independent evidence to verify that assumption. Indeed, there is strong evidence to suggest that the curvature of perceptual space varies with position (Indow, 1991; Koenderink et al., 2000), and that there are large individual differences among observers. For example, in one series of experiments by Battro et al. (1976), involving more than 120 observers, the results obtained for 60% revealed a negative curvature, whereas those for the remaining 40% revealed a positive curvature.

One possible explanation for these large variations in the metric structure of perceptual space is that the underlying geometry of observers' judgments may be dependent on contextual factors (see Suppes, 1977), such as the presence of visible objects in the environment or how an observer interprets an experimenter's instructions. We believe it is best to be skeptical, however, about the generality of this phenomenon. Although context may be important for the metric structure of perceptual space, perhaps there are other more primitive aspects of observers' perceptions that exhibit a higher degree of stability.

Within the hierarchy of possible geometries, affine structure is more primitive than Euclidean structure, because it is based on a smaller set of underlying assumptions, and is therefore invariant over a larger set of possible transformations. It is also possible to devise alternative geometries for which these assumptions are relaxed still further. For example, projective geometry is a generalization of affine geometry without any axioms about parallelism.

Just as it is possible to test the internal consistency of affine judgments using Varignon's theorem, one can also explore the projective structure of perceptual space using a theorem first proven by Pappus of Alexandria around 340 A.D. The Pappus theorem provides a global constraint on how line segments in different directions must intersect one another. We have recently performed a new set of experiments to investigate observers' collinearity judgments on a visible ground surface, and the results have confirmed that the Pappus theorem is satisfied (Koenderink, van Doorn, Kappers, & Todd, 1999).

Although most of our discussion has been focused thus far on the perceived structure of a visual ground surface, the underlying theoretical issues we have considered are also relevant to all other psychological spaces. In order to employ the concepts of geometry for modeling psychological phenomena, it is necessary to know what type of geometry is most appropriate. In the present article, we have described a new technique for testing whether a psychological space has an internally consistent affine structure, and we have considered the implications of this test for other geometric properties, including metric structure. The primary constraint on the use of this technique is that subjects must be able to make bisection judgments for designated point pairs within the psychological space to be investigated. This constraint is easily satisfied for many frequently studied perceptual properties, such as color, surface orientation, or velocity of motion. However, bisection judgments may not be possible for some psychological spaces that are discretely populated (e.g., the space of semantic categories), or for those that have an unknown dimensionality (e.g., the space of possible human faces). Additional techniques will need to be developed to identify the relevant geometry in those cases.

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